Contents

Chapter 1

Numerical Series

1.1 Series Definitions :

Definition 1.1.1

Let $U_{n\in\mathbb{N}}$ be a sequence of real numbers or complex numbers, we call a series of general term (U_n) The infinite sum of $\sum_{n\geq 1} U_n$, The sequence associated with the series $\sum_{n\geq 1} U_n$ $(S_m)_{m\geq 1}$, where for any $n \in \mathbb{N}$, $S_m = \sum_{n=1}^{m} U_n$ is called the sequence of partial sums

Remark. The sum above begin by u_1, but we often begin with u_0, u_2.

Example

Some of the classical series:

- $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}, \alpha \in \mathbb{R}$ Riemanian series (Harmonic).
- $\sum_{n\geq 1} \frac{1}{n^\alpha} (\ln n)^\beta, \alpha, \beta \in \mathbb{R}$ Bertrand series.
- $\sum_{n\geq 0} q^n, q \in \mathbb{R}$ Geometric series.
- $\sum_{n\geq 1} \frac{\sin(n\beta)}{n^\alpha}$ and $\sum_{n\geq 1} \frac{\cos(n\beta)}{n^\alpha}, \alpha, \beta \in \mathbb{R}$ Abel series.

Definition 1.1.2

Let (U_n) be a sequence of real numbers (or complex numbers), and let $(S_m)_{m\geq 1}$ be the associated sequence of partial sums.
The series $\sum_{n\geq 1} U_n$ is said to be :

- **Convergent** : if the sequence (S_m) is convergent, in this case $S = \lim_{n\to\infty} S_n$ is called the sum of series $\sum_{n\geq 1} U_n$, and we write $S = \sum_{n\geq 1} U_n$.
  Moreover, the series $R_m = S - S_m = \sum_{n=m+1}^{\infty} U_n$ is called the rest of order m of the series $\sum_{n\geq 1} U_n$.
- **Divergent** : if $\sum_{n\geq 1} U_n$ is not convergent.

The nature of a series is the fact that it converge or diverges. Two series are said to have the same nature if they both converge or both diverge.

Example

Let $q \in \mathbb{R}$ and consider the series : $\sum_{n\geq 0} q^n = 1 + q + q^2 + q^3 + ...$

$$S_m = \sum_{n\geq 0} q^n = \begin{cases} \frac{1-q^{n+1}}{1-q} & \text{if } q \neq 1 \\ m+1 & \text{if } q = 1 \end{cases}$$

$$\lim_{n\to\infty} q^n = \begin{cases} \infty & \text{if } q > 1 \\ 1 & \text{if } q = 1 \\ 0 & \text{if } q \in (-1,1) \end{cases}$$

$$\lim_{n\to\infty} S_n = \begin{cases} \infty & \text{if } q \geq 1 \\ \frac{1}{1-q} & \text{if } q \in (-1,1) \\ Indefined & \text{if } q \leq -1 \end{cases}$$

Remark.

$$\sum_{n\geq 0} q^n \text{ CV} \iff q \in (-1,1)$$

Theorem 1.1.1

Let $\sum_{n\geq 1} U_n$ and $\sum_{n\geq 1} V_n$, be two numerical series then :

$$\sum_{n\geq 1} U_n \text{ and } \sum_{n\geq 1} V_n \text{ CV} \implies \sum_{n\geq 1} U_n + V_n \text{ CV}$$

$$\sum_{n\geq 1} U_n \text{ CV} \implies \sum_{n\geq 1} \lambda V_n \text{ CV } \forall \lambda \in \mathbb{R}(\lambda \in \mathbb{C})$$

$$\sum_{n\geq 1} V_n \text{ CV and } \sum_{n\geq 1} V_n \text{ DIV} \implies \sum_{n\geq 1} U_n + V_n \text{ DIV}$$

Theorem 1.1.2 (Necessary Conditions)

Let $\sum_{n\geq 1} U_n$ be a series then we have

$$\sum U_n \text{ CV} \implies \lim_{n\to\infty} U_n = 0$$

Proof. Let S_n be the associated sequence of partial sums, we have

$$S_n - S_{n-1} = U_n$$

$$\sum_{n=1}^{\infty} U_n \text{ CV} \implies (S_n) \text{ CV} \implies \lim_{n\to\infty} U_n = \lim_{n\to\infty} (S_n - S_{n-1}) = 0$$

□

Remark.
In practice we use the contra positive that is :

$$\text{if } \lim_{n\to\infty} U_n \neq 0 \implies \sum_{n\geq 1} U_n \text{ DIV}$$

The inverse Implication is false

$$\lim_{n\to\infty} U_n = 0 \implies \sum U_n \text{ CV}$$

For instance :

$$\lim_{n\to\infty} \frac{1}{n} = 0 \text{ But } \sum_{n\geq 1} \frac{1}{n} = \infty$$

Example

- $\sum_{n\geq 1} \sin(n)$ DIV since $\lim_{n\to\infty} \sin(n)$ doesn't exist
- $\sum_{n\geq 0} \frac{n}{n+1}$ DIV, since the $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$
- $\sum_{n\geq 0} e^{-n} = \sum_{n\geq 0} \left(\frac{1}{e}\right)^n = \frac{e}{e-1}$

Theorem 1.1.3 (Cauchy Sequence)

Let $\sum_{n\geq 1} U_n$ be a series

$$\sum_{n\geq 1} U_n \text{ CV} \begin{cases} \forall \varepsilon > 0 : \exists n_\varepsilon \in \mathbb{N} : \forall n, p \in \mathbb{N} \\ m > p > n_e \implies \left|\sum_{n=p}^{m} U_n\right| \leq \varepsilon \end{cases}$$

Proof. Let $(S_k)_{k\geq 1}$ be the sequence of partial sums associated with $\sum_{n\geq 1} U_n$:

$$\begin{aligned} \sum_{n=1}^{\infty} U_n \text{ CV} &\implies (S_k)_{k\geq 1} \text{ CV} \\ &\iff (S_k)_{k\geq 1} \text{ is a cauchy sequence} \\ &\iff \begin{cases} \forall \varepsilon > 0 : \exists n_\varepsilon \in \mathbb{N} \text{ st. } : \forall m, p \in \mathbb{N} \\ m > p > n_\varepsilon \; |S_m - S_p| = |\sum_{n=1}^{m} U_n - \sum_{n=1}^{p} U_n| \leq \varepsilon \end{cases} \end{aligned}$$

□

Corollary 1.1.4

Let $\sum_{n\geq 1} U_n$ be a series and let $p \in \mathbb{N}$

$$\sum_{n\geq 1} U_n \text{ CV} \implies \sum_{n\geq p} U_n \text{ CV}$$

Proof. Let $(U_m)_{m\geq 1}$ and let $(V_m)_{m\geq 1}$ be respectively the sequences of the partial sums of $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=p}^{\infty} u_n$ for $m \geq q$:

$$|U_{m+q} - U_m| = |V_{n+q} - V_n|$$
$$\left|\sum_{n=q+1}^{m+q} u_n\right| = \left|\sum_{n=m+1}^{m+q} u_n\right|$$
$$\text{So } |U_{m+q} - U_m| < \varepsilon \implies |V_{m+q} - V_m| < \varepsilon$$

□

Theorem 1.1.5 (Telescopic series)

Let (U_n) be a sequence of real numbers, then the series $\sum_{n\geq 1}(U_{n+1} - U_n)$ and the sequencehave the same nature moreover, if (U_n) converge and has l as a limit then :

$$\sum_{n\geq 1}(U_{n+1} - U_n) = l - U_1$$

Proof. Let (S_n) be the sequence of the partial sums of $\sum_{n\geq 1}^{\infty} U_{n+1} - U_n$ we have:

$$S_n = \sum_{n=1}^{m}(U_{n+1} - U_n) = (U_{n+1} - U_n) + (U_n - U_{n-1})... = U_{n+1} - U_1$$

That shows that U_n and S_n have the same nature. □

1.2 Positive series

Definition 1.2.1

Let $\sum_{n=1}^{\infty} U_n$ be a series $\sum_{n=1}^{\infty} U_n$, is said to be positive, if there exist $n_0 \in \mathbb{N}$ such that $U_n > 0$ for all $n \geq 0$.

Example

$$\sum_{n=0}^{\infty} \frac{(-1)^n + n - 3}{n^3 + 1} \text{ is a positive series although } u_0 = -2, u_1 = -\frac{3}{2}, u_2 = 0$$

$$u_n > 0, \ \forall n \geq 4$$

Theorem 1.2.1

Let $\sum_{n=1}^{\infty} U_n$ be a positive series and let $(S_m)_{m\geq 1}$ be the corresponding series of partial sums then

$$\sum_{n=1}^{\infty} U_n \text{ CV} \iff (S_m)_{m\geq 1} \text{ is upper bounded}$$

Proof.

$$\sum_{n=1}^{\infty} U_n \text{ CV} \iff (S_m)_{m\geq 1} \text{ CV} \iff (S_m) \text{ is upper bounded } S_m \text{ is increasing}$$

Indeed $S_{m+1} - S_m = U_{m+1} > 0$ □

Theorem 1.2.2

Is a theoritical result, its used to prove a theoritical excerices
Some classical series have the form

$$\sum_{n=1}^{\infty} f(n) \left(\text{as } \sum \frac{1}{n^\alpha} \qquad f(x) = \frac{1}{x^\alpha} \right)$$

The following result provide a suggicient condition for the convergence of such type of series.

Theorem 1.2.3 (Comparison with an integral)

Let $f : [1, \infty) \to \mathbb{R}^+$ be a nonincreasing continious function, then :

- $\sum_{n=1}^{\infty} f(n)$ CV $\iff$ $\int_1^{\infty} f(x)dx$ CV
- $$\int_{m+1}^{\infty} f(x)dx \leq R_m = \sum_{n=m+1}^{\infty} f(x) \leq \int_m^{\infty} f(x)dx \qquad \forall m \in \mathbb{N}$$

Proof.

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n), \forall n \in \mathbb{N}$$

$$\begin{aligned} \int_1^{m+1} f(x)dx = \sum_{n=1}^{m} \int_n^{n+1} f(x)dx \leq S_m = \sum_{n=1}^{m} f(n) &\leq f(1) + \sum_{n=2}^{m} \int_{n-1}^{n} f(x)dx \\ &= f(1) + \int_1^m f(x)dx \end{aligned}$$

□

Example

- **Rieman Series :** $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$, $\alpha \in \mathbb{R}$
  Let $f : [1, \infty) \to [0, \infty)$ with $f(x) = \frac{1}{x^{\alpha}}$

$$f \text{ is non increasing } \iff \alpha \geq 0$$

  - $\alpha = 0$ $f(x) \implies \sum f(n)$ DIV.
  - $\alpha < 0$ in this case $\lim_{n\to\infty} f(n) \neq 0 \implies \sum f(n)$ DIV.
  - $\alpha > 0$ In this case f is decreasing

$$\sum_{n=1}^{\infty} f(n) \text{ CV } \iff \int_{1}^{\infty} \frac{1}{x^{\alpha}} dx \iff \alpha > 1$$

  - **Conclusion :**

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \iff \alpha > 1$$

- **Bertrand series :**

$$f(x) = \frac{1}{x^{\alpha}(\ln x)} \qquad f : [2, \infty) \to (0, \infty)$$

$$\begin{aligned} f \text{ is non decreasing } &\iff \alpha > 0 \text{ and } \alpha = 0 \text{ and } \beta \leq 0 \\ \lim_{n\to\infty} f(n) = 0 &\iff \alpha > 0 \text{ or } \alpha = 0 \text{ and } \beta > 0 \\ \sum f(n) \text{ CV } &\iff \int_{1}^{\infty} \frac{dx}{x^{\alpha}(\ln x)^{\beta}} \iff \alpha > 1 \text{ or } \alpha = 1 \text{ and } \beta > 1 \end{aligned}$$

  - **Conclusion :**

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}(\ln n)^{\beta}} \text{ CV } \iff \alpha > 1 \text{ or } \alpha = 1 \text{ and } \beta > 1$$

Theorem 1.2.4 (Comparison by inequality)

Let $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ be two positive series and suppose that there exist $n_0 \in \mathbb{N}$ such that

$$U_n \leq V_n \qquad \forall n \geq n_0$$

then

$$\begin{aligned} \sum_{n=1}^{\infty} \text{ CV } &\implies \sum_{n=1}^{\infty} U_n \text{ CV} \\ \sum_{n=1}^{\infty} U_n \text{ DIV } &\implies \sum_{n=1}^{\infty} V_n \text{ DIV} \end{aligned}$$

Proof. Let (S_m) and σ_m be the sequences of partial sums associated with respectively

$\sum_{n=n_0}^{\infty} U_n$ and $\sum_{n=n_0}^{\infty} V_n$.

$$\sum_{n=1}^{\infty} V_n \text{ CV } \iff \sum_{n=n_0}^{\infty} \text{ CV } \iff (\sigma_m)_{m\geq n_0} \text{ is upper bound}$$

$$\implies (S_m)_{m\geq n_0} \text{ is upper bound}$$

$$\iff \sum_{n=n_0}^{\infty} U_n \text{ CV}$$

$$\iff \sum_{n=1}^{\infty} U_n \text{ CV}$$

□

Example

- $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

$$\frac{1}{n^2+1} \leq \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ CV}$$

$$\implies \sum_{n=0}^{\infty} \frac{1}{n^2+1} \text{ CV}$$

- $\sum_{n=0}^{\infty} e^{-n^2}$

$$e^{-n^2} \leq e^{-n} = (\frac{1}{e})^n$$

$$\sum (\frac{1}{e})^n \text{ CV } \implies \sum e^{-n^2} \text{ CV}$$

Corollary 1.2.5 (Comparison by inequalities)

Let $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ be two positive series and suppose that there exist $a > 0$ and $b > 0$ $n_0 \in \mathbb{N}$ such that

$$a \leq \frac{U_n}{V_n} \leq b, \forall n \geq n_0$$

then the series $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ have the same nature.

Proof. we have

$$aV_n \leq U_n \leq bV_n, \forall n \geq n_0$$

we conclude by appliying theorem 1.2.4. □

Corollary 1.2.6 (Comparison by equivalence)

Let $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ be two positive series, then :

$$U_n \sim_{\infty} V_n \implies \sum U_n \text{ and } \sum V_n \text{ have the same nature.}$$

Proof. $U_n \sim_{\infty} V_n \iff \lim_{n\to\infty} \frac{U_n}{V_n} = 1$
Let ε_0 chosen in $(0, 1)$, by the definition of the limit there is $n_0 \in \mathbb{N}$ such that

$$0 < l - \varepsilon_0 \leq \frac{U_n}{V_n} \leq l + \varepsilon_0 \qquad \forall n \geq n_0$$

By the Carrolary 1.2.5, $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ have the same nature □

Example

- $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ DIV

$$\sin(\frac{1}{n}) \sim_{\infty} \frac{1}{n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ DIV}$$

- $\sum_{n=1}^{\infty} \frac{n+\sin(n)+1}{n^3}$ CV

$$\frac{n+\sin(n)+1}{n^3} \sim_{\infty} \frac{1}{n^2} \text{ CV}$$

Corollary 1.2.7 (Riemann Criterion)

Let $\sum_{n=1}^{\infty} U_n$ be a positive series, and suppose there is $\alpha \in \mathbb{R}$ such that $\lim_{n\to\infty} n^{\alpha} U_n = l$ then :

- If $l \in [0, \infty)$, and $\alpha > 1$, then $\sum_{n=1}^{\infty} U_n$ CV
- If $l \in (0, \infty)$ or $l = \infty$ and $\alpha \leq 1$, then $\sum_{n=1}^{\infty} U_n$ DIV.

Remark. (Reminders)

$$U_n \text{ CV} \iff (S_m) \text{ Bounded}$$

$$\frac{1}{n^{\alpha}(\ln n)^{\beta}} \quad (\alpha > 1) \text{ or } (\alpha = 1, \beta > 1)$$

Theorem 1.2.8 (Logarithmic Comparison)

Let $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ be two positive series and suppose that there is $n_0 \in \mathbb{N}$ such that :

$$\frac{U_{n+1}}{U_n} \leq \frac{V_{n+1}}{V_n} \qquad n > n_0$$

then

$$\sum_{n=1}^{\infty} V_n \text{ CV} \implies \sum_{n=1}^{\infty} U_n \text{ CV}$$
$$\sum_{n=1}^{\infty} U_n \text{ DIV} \implies \sum_{n=1}^{\infty} V_n \text{ DIV}$$

Proof. For $n > n_0$:

$$\frac{U_n}{U_{n_0}} = \frac{U_n}{U_{n-1}} \cdot \frac{U_{n-1}}{U_{n-2}} \cdots \frac{U_{n_0+1}}{U_{n_0}} \leq \frac{V_n}{V_{n-1}} \cdot \frac{V_{n-1}}{V_{n-2}} \cdots \frac{V_{n_0+1}}{V_{n_0}} = \frac{V_n}{V_{n_0}}$$

$$\implies U_n \leq \left(\frac{U_{n_0}}{V_{n_0}}\right) V_n$$

Conclusion follows from thoerem 1.2.4. □

Theorem 1.2.9 (D'almbert criterion)

Let $\sum_{n=1}^{\infty} U_n$ be a positive series such that :

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$$

Then :

$$\begin{cases} l < 1 & \implies \sum_{n=1}^{\infty} U_n \text{ CV} \\ l > 1 & \implies \sum_{n=1}^{\infty} U_n \text{ DIV} \end{cases}$$

Proof.
- Suppose $l > 1$, let $\varepsilon > 0$ s.t $: l - \varepsilon > 1,$ set $V_n = (l-\varepsilon)^n$

$$\left| \frac{U_{n+1}}{U_n} - l \right| < \varepsilon$$

$$\begin{aligned} &\implies l - \varepsilon < \frac{U_{n+1}}{U_n} < l + \varepsilon \\ &\implies \frac{V_{n+1}}{V_n} < \frac{U_{n+1}}{U_n} < l + \varepsilon \\ &\implies V_n \text{ DIV} \end{aligned}$$

since $\frac{V_{n+1}}{V_n} = l - \varepsilon > 1$, $\sum V_n$ DIV, and from theorem 1.2.8, it gives that $\sum U_n$ DIV.

- Suppose now that $l < 1$ and let $\varepsilon > 0$ be such that $l + \varepsilon < 1$, set $V_n = (l+\varepsilon)^n$, we know that $|\sum V_n|$ Converges, for such a real $\varepsilon > 0$, there exist a natural numbers $\exists n_0 \in \mathbb{N}$ such that :

$$l - \varepsilon \le \frac{U_{n+1}}{U_n} \le l + \varepsilon = \frac{V_{n+1}}{V_n}$$

Conclusion follows from theorem 1.2.8

□

Example

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$, $\quad U_n = \frac{n!}{n^n}$

$$\begin{aligned} \frac{U_{n+1}}{U_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n \\ &= \left(1 - \frac{1}{n}\right)^n \to \frac{1}{e} < 1 \end{aligned}$$

So by d'almbert criterion : $\sum U_n$ CV.

Theorem 1.2.10 (Cauchy Criterion)

Let $\sum_{n=1}^{\infty} U_n$ be a positive series and suppose that :

$$\lim_{n\to\infty} \sqrt[n]{U_n} = l \text{ then :}$$

$$l < 1 \implies \sum_{n=1}^{\infty} U_n \text{ CV}$$

$$l > 1 \implies \sum_{n=1}^{\infty} U_n \text{ DIV}$$

Proof. For arbitrary $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that :

$$\left| \sqrt[n]{U_n} - l \right| < \varepsilon$$
$$\implies l - \varepsilon < \sqrt[n]{U_n} < l + \varepsilon$$
$$\implies (l-\varepsilon)^n < U_n < (l+\varepsilon)^n$$

We conclude by theorem 1.2.4.

$$\begin{cases} l > 1 & l - \varepsilon > 1 \implies \sum U_n \text{ DIV} \\ l < 1 \text{ (and let } \varepsilon \text{ be such that)} & l - \varepsilon < 1 \implies \sum U_n \text{ CV} \end{cases}$$

□

Example

$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 + \frac{a}{n}\right)^{n^2}, \quad a \in \mathbb{R}$

$$\sqrt[n]{U_n} = \frac{1}{n^{\frac{2}{n}}} \left(1 + \frac{a}{n}\right)^n \to_\infty e^a$$

$$\begin{cases} \text{if } a < 1 & \sum U_n \text{ DIV} \\ \text{if } a > 1 & \sum U_n \text{ CV} \end{cases}$$

Corollary 1.2.11 (Comments)

Let $\sum_{n \geq 1} U_n$ be a positive series, then

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l \implies \lim_{n\to\infty} \sqrt[n]{U_n} = l$$

$$\text{(Ratio Test)} \implies \text{(Root Test)}$$

Proof. Indeed, for $\varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t :

$$l - \varepsilon < \frac{U_{n+1}}{U_n} < l + \varepsilon$$

for n large enough :

$$(l-\varepsilon)^{n-n_0+1} \leq \frac{U_{n+1}}{U_n} \cdots \frac{U_{n_0+1}}{U_{n_0}} \leq (l+\varepsilon)^{n-n_0+1}$$
$$(l-\varepsilon)^{n-n_0+1} \leq \frac{U_{n+1}}{U_n} \leq (l-\varepsilon)^{n-n_0+1}$$
$$(l-\varepsilon)^{n-n_0+1} U_{n_0} \leq U_{n+1} \leq (l-\varepsilon)^{n-n_0+1} U_{n_0}$$
$$(l-\varepsilon)^{\frac{n-n_0+1}{n+1}} U_{n_0}^{\frac{1}{n+1}} \leq U_{n+1}^{\frac{1}{n+1}} \leq (l-\varepsilon)^{\frac{n-n_0+1}{n+1}} U_{n_0}^{\frac{1}{n+1}}$$
$$\implies l - \varepsilon \leq \underline{\lim}_{n\to\infty} \sqrt[n]{U_n} \leq \overline{\lim}_{n\to\infty} \sqrt[n]{U_n} \leq l + \varepsilon$$

Since ε is arbitrary, we conclude that :

$$\lim_{n\to\infty} \sqrt[n]{U_n} = \underline{\lim}_{n\to\infty} \sqrt[n]{U_n} = \overline{\lim}_{n\to\infty} \sqrt[n]{U_n} = l$$

This ends the proof. □

Remark.

- Inverse implication is not true.
  - Take U_n as a counter example :

$$U_n = \begin{cases} 2^l \cdot 3^l & n = 2l \\ 2^l \cdot 3^{l+1} & n = 2l+1 \end{cases} \implies \begin{cases} \sqrt[n]{U_n} \to 6 \\ \lim_{n\to\infty} \frac{U_{n+1}}{U_n} \text{ Doesnt exist.} \end{cases}$$

- if $\overline{\lim}_{n\to\infty} \frac{U_{n+1}}{U_n} < 1$ or $\overline{\lim}_{n\to\infty} \sqrt[n]{U_n} < 1 \implies \lim_{n\to\infty} \sum U_n$ CV.
- if $\underline{\lim}_{n\to\infty} \frac{U_{n+1}}{U_n} > 1$ or $\underline{\lim}_{n\to\infty} \sqrt[n]{U_n} > 1 \implies \sum V_n$ DIV.
- $\overline{\lim}_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \left(\sup\left\{\frac{U_{k+1}}{U_k} : k \geq n\right\}\right)$

Theorem 1.2.12 (Raabe-Duhamel)

Let $\sum_{n\geq 1} U_n$ be a positive series such that :

$$\frac{U_{n+1}}{U_n} = 1 - \frac{l}{n} + o(\frac{1}{n}) \text{ near } \infty$$

Where $l \in \mathbb{R}$, then :

- if $l > 1$, then the series $\sum_{n\geq 1} U_n$ CV.
- if $l < 1$, then the series $\sum_{n\geq 1} U_n$ DIV.

Proof. Consider the case $l > 1$, and let $\alpha \in (1, l)$, and let $V_n = \frac{1}{n^\alpha}$

$$\frac{V_{n+1}}{V_n} = \left(\frac{n}{n+1}\right)^\alpha = \left(1 + \frac{1}{n}\right)^{-\alpha} =^\infty 1 - \frac{\alpha}{n} + o(\frac{1}{n})$$

near ∞ we have :

$$\frac{V_{n+1}}{V_n} - \frac{U_{n+1}}{U_n} = \frac{l-\alpha}{n} + o(\frac{1}{n})$$

This means that :

$$\lim_{n\to\infty} n\left(\frac{V_{n+1}}{V_n} - \frac{U_{n+1}}{U_n}\right) = l - \alpha > 0 \implies \frac{V_{n+1}}{V_n} > \frac{U_{n+1}}{U_n}$$

Using the logarithmic comparson, $\sum U_n$ CV.
Similarly if $l < 1$ we take $\beta \in (l, 1)$ and $V_n = \frac{1}{n^\beta}$ we obtain :

$$\frac{V_n}{V_n} - \frac{U_n}{U_{n+1}} = \frac{l-\beta}{n} + o(\frac{1}{n})$$

we conclude using the same way that $\sum U_n$ DIV. $\square$

Example

- $\sum_{n\geq 1} \frac{1}{n^\alpha}$, $U_n = \frac{1}{n^\alpha}$

$$\frac{U_{n+1}}{U_n} = \left(\frac{n}{n+1}\right)^\alpha = 1 - \frac{\alpha}{n} + o(\frac{1}{n})$$

By Raabe-Duhamel criterion :

$$\begin{cases} \text{if } l > 1 & \sum U_n \text{ CV} \\ \text{if } l < 1 & \sum U_n \text{ DIV} \end{cases}$$

- $\sum U_n$ with $U_n = \frac{n!}{(a+1)(a+2)(a+3)\cdot(a+n)} a \in (0, \infty)$

$$\begin{aligned} \frac{U_{n+1}}{U_n} = \frac{n+1}{(a+n+1)} &= \frac{1+\frac{1}{n}}{1+\frac{a+1}{n}} \\ &= (1+\frac{1}{n})(1+\frac{a+1}{n})^{-1} = (1+\frac{1}{n})(1-\frac{a+1}{n}+o(\frac{1}{n})) \\ &= 1 - \frac{a}{n} + o(\frac{1}{n}) \end{aligned}$$

By Raabe-Duhamel criterion :

$$\begin{cases} \text{if } l > 1 & \sum U_n \text{ CV} \\ \text{if } l < 1 & \sum U_n \text{ DIV} \end{cases}$$

Theorem 1.2.13 (Gauss Theorem)

Let $\sum_{n\geq 1} U_n$ be a positive series and suppose that there s $\alpha > 1$ and $l \in \mathbb{R}$ such that :

$$\frac{U_{n+1}}{U_n} = 1 - \frac{l}{n} + O(\frac{1}{n^\alpha}) \text{ at } \infty$$

then :

$$\begin{cases} \text{if } l > 1 & \sum U_n \text{ CV} \\ \text{if } l < 1 & \sum U_n \text{ DIV} \end{cases}$$

Remark.

$$f(x) = O(g(x)) \text{ at } x_0 \iff \frac{f(x)}{g(x)} \text{ is bounded near } x_0$$

Proof. Let $V_n = n^2 U_n$:

$$\frac{V_{n+1}}{V_n} = \left(\frac{n+1}{n}\right)^2 \frac{U_{n+1}}{U_n} = \left(1+\frac{1}{n}+O(\frac{1}{n^2})\right)\left(1-\frac{l}{n}+O(\frac{1}{n^\alpha})\right)$$

$$\begin{cases} 1+\frac{l}{n}-\frac{l}{n} & +O(\frac{1}{n^2}) & \text{if } \alpha \geq 2 \\ 1+\frac{l}{n}-\frac{l}{n} & +O(\frac{1}{n^\alpha}) & \text{if } \alpha < 2 \end{cases}$$

$$\frac{V_{n+1}}{V_n} = 1 + O(\frac{1}{n^\beta}) \text{ with } \beta = min(2,\alpha)$$

$$\ln\left(\frac{V_{n+1}}{V_n}\right) = \ln\left(1+O(\frac{1}{n^\beta})\right) = O(\frac{1}{n^\beta})$$

$\implies \ln\left(\frac{V_{n+1}}{V_n}\right) \leq \frac{M}{n^\beta} \quad \forall n > n_0 \quad M > 0$ The series $\sum \ln V_{n+1} - \ln V_n$ CV.

$$S = \sum \ln V_{n+1} - \ln V_n = \lim_{n\to\infty} S_m$$
$$= \lim_{n\to\infty} \sum \ln V_{n+1} - \ln V_n$$
$$\implies \lim_{n\to\infty} \ln V_{n+1} = S + \ln V_1 = k$$
$$\lim_{n\to\infty} V_{n+1} = e^k$$

Conclusion : $\lim_{n\to\infty} n^2 U_n = e^k \implies U_n \sim \frac{e^k}{n^2}$ at ∞. ☐

Example

- $\sum \frac{1}{n^\alpha}, \quad \frac{1}{n^\alpha} = U_n$

$$\frac{U_{n+1}}{U_n} = \left(1 + \frac{1}{n}\right)^{-\alpha} = 1 - \frac{\alpha}{n} + O(\frac{1}{n^2})$$

By Gauss Criterion :

$$\begin{cases} \text{if } \alpha > 1 \text{ then } \sum \frac{1}{n^\alpha} \text{ CV} \\ \text{if } \alpha < 1 \text{ then } \sum \frac{1}{n^\alpha} \text{ DIV} \end{cases}$$

- $\sum U_n \quad U_n = \frac{n! e^n}{n^{n+p}} \quad p \in \mathbb{R}$

HINT :

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)! e^{n+1}}{(n+1)^{n+1+p}} \frac{n^{n+p}}{n! e^n} = e\left(\frac{n}{n+1}\right)^{n+p} = e\left(1 + \frac{1}{n}\right)^{-(n+p)}$$
$$= \left(1 + \frac{1}{n}\right)^{-(n+p)} = e^{-(n+p)\ln\left(1+\frac{1}{n}\right)}$$

1.3 Alternating Series

An alternating series is a series whose general term, changes sign infinitely many times

Example

The series $\sum_{n=1}^{\infty} n \sin n$ is an alternating series.

Definition 1.3.1

The series $\sum_{n=1}^{\infty} U_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |U_n|$ is convergent.

Corollary 1.3.1

If a series converges absolutely, it converges.

Proof. Let $\sum_{n=1}^{\infty} U_n$ be a series converging absolutely ($\sum_{n=1}^{\infty} |U_n|$) by theorem 1.1.5, $\forall \varepsilon > 0, \quad \exists n_\varepsilon \in \mathbb{N}$, such that for all $m, p \in \mathbb{N}$:

$$m \geq n_\varepsilon \implies \sum_{n=m+1}^{m+p} |U_n| \leq \varepsilon$$

But since, $\left|\sum_{n=m+1}^{m+p} U_n\right| \leq \sum_{n=m+1}^{m+p} |U_n|$, for all $m \geq n_\varepsilon$, and for all $p \in \mathbb{N}$ we have :

$$\left|\sum_{n=m+1}^{m+p} U_n\right| \leq \varepsilon$$

Hence, $\sum_{n=1}^{\infty} U_n$ converges. $\square$

Example

- $\sum_{n=1}^{\infty} U_n$, with $U_n = \frac{\cos n}{n^2}$.

$$\left|U_n = \frac{|\cos n|}{n^2}\right| \leq \frac{1}{n^2}$$

$$\text{Since } \sum \frac{1}{n^2} \text{ CV } \implies \sum \frac{\cos n}{n^2} \text{ CV Absolutely}$$
$$\implies \sum_{n=1}^{\infty} \frac{\cos n}{n^2} \text{ CV}$$

- $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{n}+(-1)^n}$

$$\left|\frac{(-1)^n}{n\sqrt{n}+(-1)^n}\right| = \frac{|(-1)^n|}{|n\sqrt{n}+(-1)^n|} = \frac{1}{n\sqrt{n}+(-1)^n}$$

$$\sim_\infty \frac{1}{n\sqrt{n}} \quad \left(\frac{1}{n\sqrt{n}+(-1)^n} = \frac{1}{n\sqrt{n}}\left(\frac{1}{1+\frac{(-1)^n}{n\sqrt{n}}}\right)\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \text{ CV } \implies \sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n\sqrt{n}+(-1)^n}\right| \text{ CV } \implies \sum \frac{(-1)^n}{n\sqrt{n}+(-1)^n} \text{ CV}$$

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \text{ DIV}$$

Theorem 1.3.2 Leibniz

Consider the series $\sum_{n=1}^{\infty}(-1)^n a_n$ If (a_n) is an non increasing having $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Proof. Let $(S_m)_{m\geq 1}$ be the sequence of partial sums associated with $\sum_{n=1}^{\infty}(-1)^n a_n$.

$$\begin{aligned} S_{2m+2} - S_{2m} &= \sum_{n=1}^{2m+2}(-1)^n a_n - \sum_{n=1}^{2m}(-1)^n a_n \\ &= (-1)^{2m+1}a_{2n+1} + (-1)^{2m+2}a_{2m+2} = a_{2m+2} - a_{2m+1} \leq 0 \end{aligned}$$

$(S_{2m})m \geq 1$ is non increasing.

$$\begin{aligned} S_{2m+3} - S_{2m+1} &= (-1)^{2m+2}a_{2m+2} + (-1)^{2m+3}a_{2m+3} = a_{2m+2} - a_{2m+3} \geq 0 \\ S_{2m+1} - S_{2m} &= (-1)^{2m+1}a_{2m+1} \to 0 \,(\text{ as } m \to \infty) \end{aligned}$$

Conclusion (S_{2m}) and (S_{2m+1}) are adjacent, therefore (S_m) converges, that is $\sum_{n=1}^{\infty}(-1)^n a_n$ CV. $\square$

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ CV} \iff \alpha > 0$$

Indeed,

$$\begin{cases} \text{If } \alpha \leq 0 \quad \lim_{n\to\infty} \frac{(-1)^n}{n^{al}} \neq 0 \text{ (Does not exist)} \\ \text{If } \alpha > 0 \text{ we have } \left(\frac{1}{n^\alpha}\right) \text{ is decreasing and } \lim_{n\to\infty} \frac{1}{n^\alpha} = 0 \end{cases}$$

Theorem 1.3.3 (Abel's Criterion

Let $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ be two real sequences, if the following conditions are satisfied.

- $\exists M > 0, \quad |\sum_{n=1}^{m} U_n| \leq M \quad \forall m \in \mathbb{N}$
- $\sum_{n=1}^{\infty} |V_n - V_{n+1}|$ CV.
- $\lim_{n\to\infty} V_n = 0$

Then :

$$\sum_{n=1}^{\infty} U_n V_n \text{ CV}$$

Proof. For any $m, p \in \mathbb{N}$, Let $S_m = \sum_{n=m+1}^{m+p} V_n$

$$\begin{aligned} \left|\sum_{n=m+1}^{m+p} U_n V_n\right| &= \left|\sum_{n=m+1}^{m+p} (S_n - S_{n-1}) V_n\right| \\ &= \left|\sum_{n=m+1}^{m+p} S_n V_n - \sum_{n=m+1}^{m+p} S_{n-1} V_n\right| \\ &= \left|\sum_{n=m+1}^{m+p} S_n V_n - \sum_{n=m}^{m+p-1} S_n V_{n+1}\right| \\ &= \left|\sum_{n=m+1}^{m+p-1} S_n (V_n - V_{n+1}) - S_m V_{m+1} + S_{m+p} V_{m+p}\right| \\ &\leq M \left(\sum_{n=m+1}^{m+p-1} |V_n - V_{n+1}| + |V_{m+1}| + |V_{m+p}|\right) \end{aligned}$$

Let $\varepsilon > 0$, since $\lim_{n\to\infty} V_n = 0$, and $\sum_{n=1}^{\infty} |V_n - V_{n+1}|$ CV, there is $n_\varepsilon \in \mathbb{N}$:

$$\sum_{n=m+1}^{m+p-1} |V_n - V_{n+1}| \leq \frac{\varepsilon}{3M} \quad \forall m \geq n_\varepsilon \quad \forall p \in \mathbb{N}$$

$$\implies |V_n| \leq \frac{\varepsilon}{3M} \quad \forall n \geq n_\varepsilon$$

Hence for $m \geq n_\varepsilon$:

$$\begin{aligned} \sum_{n=m+1}^{m+p-1} |S_n V_n| &\leq M \left(\sum_{n=m+1}^{m+p-1} |V_n - V_{n+1}| + |V_{m+1}| + |V_{m+p}|\right) \\ &\leq M \left(\frac{\varepsilon}{3M} + \frac{\varepsilon}{3M} + \frac{\varepsilon}{3M}\right) = \varepsilon \end{aligned}$$

The proof is complete. □

Example

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha}, \quad \sum_{n=1}^{\infty} \frac{\sin nx}{n^\alpha}, \quad x \in \mathbb{R}, \alpha > 0$$

Set $U_n = \cos nx$ (resp. $U_n = \sin nx$), and $V_n = \frac{1}{n^\alpha}$

- $\lim_{n\to\infty} V_n = 0 \quad (\alpha > 0)$
- $|V_n - V_{n+1}| = V_n - V_{n+1} = \frac{1}{n^\alpha - \frac{1}{(n+1)^\alpha}} = \frac{1}{n^\alpha}\left(1 - \frac{1}{\left(1+\frac{1}{n}\right)^\alpha}\right) \sim_\infty \frac{\alpha}{n^{\alpha+1}}$
  and $\sum \frac{1}{n^\alpha} \to$ CV $(\alpha + 1 > 1)$, so it converges.
- $$\begin{aligned} \cos nx = &\ \mathcal{R}\left(e^{inx}\right) \\ \sin nx = &\ \mathcal{I}\left(e^{inx}\right) \end{aligned}$$

  $$\left|\sum_{n=0}^{m} \cos nx\right| = \mathcal{R}\left(\sum_{n=0}^{m} e^{inx}\right)$$
  $$\left|\sum_{n=0}^{m} \sin nx\right| = \mathcal{I}\left(\sum_{n=0}^{m} e^{inx}\right)$$

  $$\left|\sum_{n=0}^{m} e^{inx}\right| = \left|\sum_{n=0}^{m} \left(e^{ix}\right)^n\right| = \left|\frac{1 - e^{i(m+1)x}}{1 - e^{ix}}\right| = \frac{\left|1 - e^{-ix} - e^{i(m+1)x} + e^{imx}\right|}{\left|1 - e^{ix}\right|} \leq \frac{4}{1 - e^{ix}} = M$$

  Conclusion :

  $$\sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha} \text{ and } \sum_{n=1}^{\infty} \frac{\sin nx}{n^\alpha} \text{ CV } \iff \alpha > 0$$

Use of Asymptotic Development :

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n} \text{ set } U_n = \frac{(-1)^n}{\sqrt{n} + (-1)^n}$$

$$\begin{aligned} U_n &= \frac{(-1)^n}{\sqrt{n}\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)} = \frac{\frac{(-1)^n}{\sqrt{n}}}{1 + \frac{(-1)^n}{\sqrt{n}}} \\ &= \frac{x}{1+x} \end{aligned}$$

$$f(x) = \frac{x}{1+x} = x - x^2 + x^3 + o(x^3) \text{ near } 0$$

$$U_n = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + \frac{(-1)^n}{n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ CV by Leibneiz}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ DIV}$$

$$\sum \frac{(-1)^n}{n\sqrt{n}} \text{ CV}$$

$\sum_{n=1}^{\infty} o\left(\frac{1}{n\sqrt{n}}\right)$, CV absolutely so CV

Indeed $\lim_{n\to\infty} \frac{\left|o\left(\frac{1}{n\sqrt{n}}\right)\right|}{\frac{1}{n\sqrt{n}}} = 0 \implies \left|o\left(\frac{1}{n\sqrt{n}}\right)\right| \leq \frac{M}{n\sqrt{n}}$

Chapter 2

Sequences of functions

2.1 Generalities

In all this chapter, we let I be an interval and we denote by $\mathcal{F}(I, \mathbb{R})$ the set of real function defined on I.
For any bounded function $f \in \mathcal{F}(I, \mathbb{R})$, the symbol $||f||$ denoted the sequence of $|f|$ on I, that is :

$$||f|| = sup_{x \in I} |f(x)|$$

Definition 2.1.1

We call a sequence of functions any mapping $\mathbb{N} \to \mathcal{F}(I, \mathbb{R})$, usually a sequence of functions is denoted by $(f_n)_{n \geq 0}$, $(g_n)_{n \geq 1}$ $\cdots$

Example

- $I = [0, 1], \quad f_n(x) = x^n$
- $I = \mathbb{R}, \quad g_n(x) = e^{nx}$

Definition 2.1.2

let $(f_n)_{n \geq 1} \subset \mathcal{F}(I, \mathbb{R})$ be a sequence of functions, we say that $(f_n)_{n \geq 1}$ is point wise convergent to $f \in \mathcal{F}(I, \mathbb{R})$ on I, if for all $x \in I$ we have

$$\lim_{n \to \infty} f_n(x) = f(x)$$

Pointwise convergence defines the converges of a function in term of their values of their domains, we say that a sequence $(f_n)_{n \geq 1}$ is pointwise convergent if it converges to some functions.

Example

- $(f_n)_{n\geq 1}$ defined by $f_n(x) = x^n, \quad x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 \text{ if } x \in (0,1) \\ 1 \text{ if } x = 1 \end{cases}$$

- $f_n(x) = \frac{x}{n} \quad x \in \mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = 0$$

- $f_n(x) = 1 + e^{-nx} \quad x \in [0,\infty)$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 2 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$$

Pointwise converges is the netural way to define the convergence of a seqeucnce of af unctions? Unfortunately, this of convergence doesnt preserve certain properties of the sequence, the following examples illustrate this situation.

Example

Let $(f_n)_{n\geq 1}$, be the sequence defnied on $(0,\pi/2)$ by $f_n(x) = \frac{nx}{nx^2+\cos x}$
We have

$$\lim_{n\to\infty} f_n(x) = \frac{1}{x}$$

Note that for all $n \in \mathbb{N}$

$$0 \leq f_n(x) \leq \frac{n\frac{\pi}{2}}{nx^2 + \cos x} = g_n(x)$$

$$g_n'(x) = \frac{n\frac{\pi}{2} - (2xn - \sin x)}{(nx^2+\cos x)^2} \leq 0 \implies g_n(x) \leq \frac{n\pi}{2}, \forall x \in (0, \frac{\pi}{2})$$

For all $n \in \mathbb{N}$, f_n is bounded and continious, In particular, f_n is intergrable on $[0,1]$, But f is not bounded ($\lim_{x\to 0} f_n(x) = \infty$) and f is not integrable.

Example

$$f_n(x) = \frac{x^2}{\sqrt{x^2 + \frac{1}{n}}}, \quad x \in \mathbb{R}, \quad \lim_{n\to\infty} f_n(x) = |x|$$

For all $n \in \mathbb{N}$, f_n is diffirentiable at 0 but f is not diffirentiable at 0.

2.2 Uniform Convergence

In this section, we introduce the mode of convergence stronger than pointwise one, the diffirence between the two modes is analogous to that of uniform continiouty.

Definition 2.2.1

Let $(f_n)_{n\geq 1} \subset \mathcal{F}(I, \mathbb{R})$ be a sequence of functions and let $f \in \mathcal{F}(I, \mathbb{R})$ we say that $(f_n)_n$ is uniformaley convergent to f on I , and we write $f_n \to^U f$ on I, if for all $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$n \geq n_\varepsilon \implies |f_n(x) - f(x)| \leq \varepsilon \quad \forall x \in I$$

Remark. Notice that a sequence $(f_n)_{n\geq 1} \subset \mathcal{F}(I, \mathbb{R})$ converges uniformalley to $f \in \mathcal{F}(I, \mathbb{R})$ if and only if

$$sup_{x\in I} |f_n(x) - f(x)| = ||f_n - f|| \to 0 \quad \text{ as } n \to \infty$$

$$\left(\lim_{n\to\infty} sup_{x\in I} |f_n(x) - f(x)|\right) = \lim_{n\to\infty} ||f_n - f|| = 0$$

Corollary 2.2.1

If a sequence of functions $(f_n)_{n\geq 1}$ converges uniformally to $f \in \mathcal{F}(I, \mathbb{R})$, then for all $x \in I$, we have $\lim_{n\to\infty} f_n(x) = f(x)$.

Proof. Easy. □

Example

$$f_n(x) = x^n \quad x \in I = [0, 1]$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in [0, 1] \end{cases} = f(x)$$

Example

$$f_n(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2n} \\ -2x + \frac{1}{n} & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = 0 \quad f_n \to 0 \quad \text{ on } I = [0, 1]$$

$$||f_n - 0|| = sup_{x\in[0,1]} |f_n(x)| = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

So $f_n \to^U 0$

Remark. Study the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ with $f_n(x) = \left(1 + \frac{x}{n}\right)^n$ on $\mathbb{R}$.

Theorem 2.2.2 (Cauchy)

let $(f_n)_{n\geq 1} \subset \mathcal{F}(I,\mathbb{R})$ be a sequence of functions, $f \in \mathcal{F}(I,\mathbb{R})$.

$$f_n \to^U f \iff \begin{cases} \forall \varepsilon > 0, \quad \exists n_e \in \mathbb{N} \\ \forall n, m \in \mathbb{N}, \quad m, n \geq n_e \implies ||f_n - f_m|| \leq \varepsilon \end{cases}$$

Proof.

$$(\implies)$$

If $f_n \to^U f_m$,then for $\forall \varepsilon > 0, \quad \exists n_e \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$:

$$n \geq n_e \implies ||f_n - f|| \leq \varepsilon$$

Hence for $m, n \geq n_e$ we have :

$$\begin{aligned} ||f_n - f_m|| &= sup_{x\in I} |f_n(x) - f_m(x)| \\ &\leq sup_{x\in I} (|f_n - f(x)| + |f_m(x) - f(x)|) \\ &\leq sup_{x\in I} |f_n(x) - f(x)| + sup_{x\in I} |f_m(x) - f(x)| \\ &= ||f_n - f|| + ||f_m - f|| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

$$(\impliedby)$$

First, let $\varepsilon > 0, \quad \exists n_e \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}$

$$n, m \geq n_e \implies |f_n(x) - f_m(x)| \leq ||f_n - f_m|| \leq \varepsilon \quad \forall x \in I$$

This means that f_n for all $x \in I$, $(f_n(x))$ is a cauchy sequence, so it converge to some $f(x)$.
For any $\varepsilon > 0, \exists n_e \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}$.

$$|f_n(x) - f_m(x)| \leq \varepsilon \quad \forall x \in I$$

this implies that

$$\varepsilon \geq \lim_{m\to\infty} |f_n(x) - f_m(x)| \text{ abs is continious so } \implies \left|f_n(x) - \lim_{m\to\infty} f_m(x)\right|$$

$$= |f_n(x) - f(x)| \quad \forall n \geq n_e \quad \forall x \in I$$

Hence

$$||f_n - f|| = sup_{x\in I} |f_n(x) - f(x)| \leq \varepsilon \quad \forall n \geq n_e$$

That is $f_n \to^U f$ on I. □

2.3 Properties of the uniform convergence

In all this section, we let $(f_n) \subset \mathcal{F}(I,\mathbb{R})$ be a sequence of functions and $f \in \mathcal{F}(I,\mathbb{R})$.

Theorem 2.3.1 (Boundedness)

Suppose that $f_n \to^U f$ on I and there is $n_0 \in \mathbb{N}$ such that f_n is bounded on I for all $n \geq n_0$, Then f is bounded on I.

Proof. For any $\varepsilon > 0, \quad \exists n_e \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, n \geq n_e \implies (|f_n(x) - f(x)| \leq \varepsilon \quad \forall x \in I)$$

Let $n_* \geq max(n_e, n_0)$, then $\forall x \in I$

$$\begin{aligned}|f(x)| &\leq |f_{n_e}(x) - f(x)| + |f_{n_0}(x)| \\ &\leq \varepsilon + ||f_{n_*}||\end{aligned}$$

So f is bounded on I. □

Theorem 2.3.2 (Integrablity)

Suppose that $f_n \to^U f$ on I and there is $n_0 \in \mathbb{N}$ such that f_n is integrable (In Riemann sens) on $[a,b] \subset I$ for all $n \geq n_0$. Then f is integrable on $[a,b]$, and we have

$$\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b \lim_{n\to\infty} f_n(t)dt = \int_a^b f(t)dt$$

Proof. Let $\varepsilon > 0$, there is $n_e \in \mathbb{N}$ such that for all $n \geq n_e$, we have :

$$f_n(x) - \frac{\varepsilon}{4(b-a)} \leq f(x) \leq f_n(x) + \frac{\varepsilon}{4(b-a)}$$

Also, for all $n \geq n_0$, there is a subdivision $\{x_0, x_1, \ldots, x_k\}$

$$(a = x_0 < x_1 < x_2 < \ldots < x_k = b)$$

such that

$$\sum_{i=1}^{k} (M_{ni} - m_{ni})(x_i - x_{i-1}) \leq \frac{\varepsilon}{2}$$

We have

$$M_{ni} = sup_{x\in[x_{i-1},x_i]} f_n(x) \quad m_{ni} = inf_{x\in[x_{i-1},x_i]} f_n(x)$$

Let

$$M_i = sup_{x\in[x_{i-1},x_i]} f(x) \text{ and } m_i = inf_{x\in[x_{i-1},x_i]} f(x)$$

Therefore we have :

$$\begin{aligned}M_{ni} - \frac{\varepsilon}{4(b-a)} \leq M_i \leq M_{ni} + \frac{\varepsilon}{4(b-a)} \\ m_{ni} - \frac{\varepsilon}{4(b-a)} \leq m_i \leq m_{ni} + \frac{\varepsilon}{4(b-a)}\end{aligned}$$

$$\begin{aligned}S(f,(x_i)) - s(f,x_i) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \left[(M_{ni} - m_{ni}) + \frac{\varepsilon}{2(b-a)}\right](x_i x_{i-1}) \\ &= \sum_{i=1}^{n} (M_{ni} - m_{ni})(x_i - x_{i-1}) + \frac{\varepsilon}{2(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

We have proved that f is integrable.

$$\left| \int_a^b f(t)dt - \int_a^b f_n(t)dt \right|$$

$$\begin{aligned} &\leq \int_a^b |f_n(t) - f(t)|\, dt \\ &\leq \int_a^b ||f_n - f||\, dt \\ &= (b-a)\, ||f_n - f|| = 0 \end{aligned}$$

□

Corollary 2.3.3

Suppose that $f_n \to^U f$ on $[a, b] \subset I$, and $\exists n_0 \in \mathbb{N}$, such that f_n is integrable on $[a, b]$ for all $n \geq n_0$. Then $F_n \to^U F$, on $[a, b]$ where

$$F_n(x) = \int_a^x f_n(t)dt \quad \text{and} \quad F(x) = \int_x^x f(t)dt$$

Proof. For all $x \in [a, b]$ we have

$$\begin{aligned} |F_n(x) - F(x)| &\leq \int_a^x |f_n(t) - f(t)|\, dt \\ &\leq (b-a)\, ||f_n - f|| \end{aligned}$$

This implies :

$$\begin{aligned} ||F_n - F|| &= sup_{x \in [a,b]} |F_n(x) - F(x)| \\ &\leq (b-a)\, ||f_n - f|| \to 0 \text{ as } n \to \infty \end{aligned}$$

□

Theorem 2.3.4 (Permutation of limits)

Suppose that $f_n \to^U f$ on I and there is $n_0 \in \mathbb{N}$ such that $\lim_{x \to a} f_n(x) = l_n \in \mathbb{R}$ $\forall n \geq n_0$, where $a \in I$, then, $\lim_{x \to a} f(x) = l \in \mathbb{R}$, and we have

$$\lim_{n \to \infty} (\lim_{x \to a} f_n(x)) = \lim_{x \to a} (\lim_{n \to \infty} f_n(x))$$

Proof. $f_n \to^U f \implies (f_n)$ is a Cauchy sequence.
Hence, for any $\varepsilon > 0$, $\exists n_e \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}$.

$$|f_n(x) - f_m(x)| \leq \varepsilon \quad \forall x \in I \quad \forall n, m \geq n_e$$

Passing to the limit, where $x \to a$, we obtain

$$|l_n - l_m| \leq \varepsilon \quad \forall n, m \geq n_e$$

This means that (l_n) is a cauchy sequence and $l_n \to l \in \mathbb{R}$. Let $\varepsilon > 0$, there is $n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{3} \forall x \in I \quad (f_n \to^U f) \text{ on } I$$

$$\text{For all } n \geq n_e \quad \exists \delta_{n,e} > 0 \quad |x - a| \leq \delta_{n,e} \implies |f_n(x) - l_n| \leq \frac{\varepsilon}{3}$$

$$\exists n_2 \in \mathbb{N}, \quad |l_n - l| \leq \frac{\varepsilon}{3} \quad \forall n \geq n_2$$

Choosing $n_* \geq max(n_0, n_1, n_2), \exists \delta_{n_*,\varepsilon} > 0$
such that $|x - a| \leq \delta_{n_*,\varepsilon}$

$$\begin{aligned} |f(x) - l| &\leq |f(x) - f_{n_*}(x)| + |f_{n_*}(x) - l_n| + |l_n - l| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \end{aligned}$$

This shows that $\lim_{x\to a} f(x) = l$ in other words.

$$\lim_{x\to a}\left(\lim_{n\to\infty} f(x)\right) = l = \lim_{n\to\infty} l_n = \lim_{n\to\infty}\left(\lim_{x\to a} f_n(x)\right)$$

$$\lim_{x\to a} f(x) = l \quad \forall\varepsilon > 0, \exists\delta > 0, \forall x \in I, |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \varepsilon$$

$$\begin{aligned}|f(x) - l| &\leq |f_n(x) - f_n(x)| + |f_n(x) - l| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - l_n| + |l_n - l|\end{aligned}$$

□

Remark. In Theorem 2.3.4 a can be ∞ or $-\infty$.
Also, Theorem 2.3.4 holds if $\exists n_0 \in \mathbb{N}$ such that $\lim_{x\to\infty} f_n(x) = \infty$ or $-\infty$, in this case, we have $\lim_{x\to a} f(x) = \infty$ or $-\infty$.

Corollary 2.3.5 (Continiuty)

Suppose that $f_n(x) \to f$ on I, and there is $n_0 \in \mathbb{N}$ such that f_n is continious at a, where $a \in I$, then f is continious on a, in particular if f_n is continious on I for all $n \geq n_0$, then f is continious on I.

Proof.

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(\lim_{n\to\infty} f_n(x)\right) \\ &= \lim_{n\to\infty}\left(\lim_{x\to a} f_n(x)\right) \\ &= \lim_{n\to\infty} f_n(a) = f(a).\end{aligned}$$

□

Theorem 2.3.6 (Differentiability)

Suppose that there is $n_0 \in \mathbb{N}$ such that f_n is continuously differentiable on $[a, b] \subset I$, If $f_n' \to g$ uniformally on $[a, b]$ and there is $x_0 \in [a, b]$, such that $f_n(x_0)$ converges, then :

$(f_n)_{n\geq 1}$ is uniformally convergent on $[a, b]$ to some function f, f is continuously differentiable and we have $f' = g$ on $[a, b]$.

Proof. For all $n \geq n_0$, and we have :

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)dt$$

By Corollary 2.3.3, we have :

$$f_n \to^U \alpha + \int_{x_0}^{x} g(t)dt = f(x)$$

with $f'(x) = g(x)$ □

Corollary 2.3.7

Suppose that there is $n_0 \in \mathbb{N}$ such that $f_n \in C^k([a,b])$ with $[a,b] \subset I$, and $k \geq 2$. If $f_n^{(k)} \to g$ on $[a,b]$ and there is $x_0 \in [a,b]$, such that $\left(f_n^{(i)}(x_0)\right)$ converge for all $i \in \{0,1,\ldots,k-1\}$ then $\left(f_n^{(i)}\right)_{n\geq 1}$ converge uniformaly $\forall i \in \{0,1,\ldots,k-1\}$ to some $f \in C^k[a,b]$ and we have $f^{(k)} = g$.

Proof.

$$f_n^{k-1}(x) = f_n^{(k-1)}(x_0) + \int_{x_0}^{x} f_n^{(k)}(t)dt.$$

$$\implies f_n^{(k-1)} \to^U +\alpha + \int_{x_0}^{x} g(t)dt$$

By applying $k-1$ times. $\square$

Chapter 3

Series of Functions

In all this chapter, we let I be a real interval and we denote by $\mathcal{F}(I,\mathbb{R})$ the set of all real function defined on I, For $f \in \mathcal{F}(I,\mathbb{R})$ with f bounded, the symbol $\|f\|$ denotes the supremum of $|f|$ on I, that is

$$\|f\| = sup_{x\in I} |f(x)|$$

3.1 Definitions :

Definition 3.1.1

let $(f_n)_{n\geq 1} \subset \mathcal{F}(I,\mathbb{R})$ be a sequence of functions, we call the series of general term f_n, the infinite sum $\sum_{n=1}^{\infty} f_n (n \sum_{n=1}^{\infty} f_n)$, the sequence $(S_m)_{m\geq 1}$ where $S_m(x) = \sum_{n=1}^{m} f(x)$ is called the sequence of partial sums associated with the series $\sum_{n=1}^{\infty} f_n$, the series $\mathcal{R}_m = \sum_{n=m+1}^{\infty} f_n$ is called the rest of order m.

Example

1. $\sum_{n=1}^{\infty} x^n$ Geometric series
2. $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$

Definition 3.1.2

Let $(f_n)_{n\geq 1} \subset \mathcal{F}(I,\mathbb{R})$ be a sequence of functions and let $(S_m)_{m\geq 1}$ be the associated sequence of partial sums, the series $\sum_{n\geq 1} f_n$ is said to be convergent at $x_0 \in I$, if the series $\sum_{n\geq 1} f_n(x_0)$ converges.

in such situation if $\lim_{n\to\infty} S_n(x_0) = S(x_0)$ we say that the series $\sum_{n=1}^{\infty} f_n(x_0)$ then it's sum equal to $S(x_0)$ and we write

$$S(x_0) = \sum_{n=1}^{\infty} f_n(x_0)$$

The set

$$\mathcal{D} = \left\{ x \in I : \sum_{x=1} f_n x \text{ CV} \right\}$$

Is called the domain of convergence of the series $\sum_{n=1}^{\infty} f_n$

Example

1. $\sum_{n=0}^{\infty} x^n \quad D = (-1, 1) \quad S(x) = \frac{1}{1-x}$
2. $\sum_{n=0}^{\infty} (-1)^n x^n \quad D = (-1, 1) \quad S(x) = \frac{1}{1+x}$
3. $\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad D = \mathbb{R}$

$$a_n = \left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!}$$
$$\frac{a_{n+1}}{a_n} = \frac{|x|}{n+1} \to 0$$

$$\implies \sum_{n\geq 0} \left|\frac{x^n}{n!}\right| \text{ CV } \forall x \in \mathbb{R}$$
$$\implies \sum_{n\geq 0} \frac{x^n}{n!} \text{ CV } \forall x \in \mathbb{R}$$
$$\implies D = \mathbb{R}$$

3.2 Uniform and Normal Convergence

In all this section, we let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(I, \mathbb{R})$ be a sequence of function and let $(S_m)_{m\geq 1}$ the sequence of partial sums associated with the series $\sum_{n\geq 1} f_n$

Definition 3.2.1

The series of functions, $\sum_{n=1}^{\infty} f_n$ is said to uniformally convergent on I, if the sequence $(S_m)_{m\in\mathbb{N}}$ is uniformally convergent on I.

Theorem 3.2.1 Cauchy

This series $\sum_{n=1}^{\infty}$ converge uniformaly on I, if and only if

$$\forall \varepsilon > 0, \quad \exists n_e \in \mathbb{N} \text{ s.t. } \forall m, p \in \mathbb{N}$$

$$m \geq n_e \implies \left\|\sum_{n=m+1}^{m+p} f_n\right\| \leq \varepsilon$$

Proof. Easy!, yeah sure. □

Example

$\sum_{n=1}^{\infty} x^n \quad D_c = (-1, 1)$ and it's sum

$$S(x) = \frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

$$S_m(x) = \sum_{n=0}^{m} x^n \quad |S_m(x)| \leq m+1 \quad \forall x \in D_c$$

But $\lim_{m\to\infty} S_m(x) = \frac{1}{1-x}$ is not bounded.
So $(S_m)_{m\in\mathbb{N}}$ does not converge uniformally on $(-1, 1)$

$$\text{Notice } \int_{-1}^{1} (S_m)dx \text{ CV} \quad \text{and } \int_{-1}^{1} S(x)dx \text{ DIV}$$

Let $a \in (0, 1)$, we have :

$$\begin{aligned} sup_{x\in[-a,a]} |S(x) - S_m x| &= sup_{x\in[-a,a]} \left| \sum_{n=m+1}^{\infty} x^n \right| = sup_{x\in[-a,a]} \left| x^{m+1} \sum_{n=0}^{\infty} x^n \right| \\ &= sup_{x\in[-a,a]} \frac{\left|x^{m+1}\right|}{1-x} \leq a^m sup_{x\in[-a,a]} \frac{1}{1-x} = \frac{a^m}{1-a} \end{aligned}$$

So $\sum_{n=0}^{\infty} x^n$ Converge Uniformalley to $\frac{1}{1-x}$ on $[-a, a]$ for any $a \in (0, 1)$

Example

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

First we have $D_c = \mathbb{R}$, set

$$U_m(x) = \left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!}$$

$$\frac{U_{n+1}(x)}{U_n(x)} = \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty \quad \forall x \in \mathbb{R}$$

By d'almbert criterion $\sum \left|\frac{x^n}{n!}\right|$ CV $\forall x \in \mathbb{R}$, we deduce $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$, so $D_c = \mathbb{R}$,

$$\begin{aligned}\|S - S_m\| &= \sup_{x\in[-a,a]} \left|\sum_{n=m+1}^{\infty} \frac{x^n}{n!}\right| \\ &= \sup_{x\in[-a,a]} \sum_{n=m+1}^{\infty} \frac{|x|^n}{n!} \le \sum_{n=m+1}^{\infty} \frac{|a|^n}{n!} = |S(|a|) - S_m(|a|)| \to 0, m \to \infty\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^x} \qquad D_c = (1, \infty)$$

Let us show that it converges uniformalley on $[a, \infty)$ with $a > 1$

$$\begin{aligned}\|S - S_m\| &= \sup_{x\in[a,\infty)} \left|\sum_{n=m+1}^{\infty} \frac{1}{n^x}\right| = \sup_{x\in[a,\infty)} \sum_{n=m+1}^{\infty} \frac{1}{n^x} \\ &= \sup_{x\in[a,\infty)} \sum \exp(-x \ln n) \le \sum_{n=m+1}^{\infty} \exp(-a \ln n) = \sum_{n=m+1}^{\infty} \frac{1}{n^a} \\ &= |S(a) - S_m(a)| \to 0 \text{ as } n \to \infty\end{aligned}$$

So $\sum_{n=1}^{\infty} \frac{1}{n^x}$ CV uniformaley on $[a, \infty)$.

Definition 3.2.2

We say that the series $\sum_{n=1}^{\infty} (f_n)$ converges normally on I, if

$$\sum_{n=1}^{\infty} \|(f_n)\| \text{ CV}$$

Corollary 3.2.2

Let $\sum (f_n)$ be a series of function, then we have :

$$\sum_{n=1}^{\infty} (f_n) \text{ CV Normally } I \implies \sum_{n=1}^{\infty} (f_n) \text{ CV Uniformally on } I$$

Proof. For any $m, p \in \mathbb{N}$, we have

$$\begin{aligned}\| \sum_{n=m+1}^{m+p} (f_n) \| = \sup_{x \in I} \left| \sum_{n=m+1}^{m+p} f_n(x) \right| &\leq \sup_{x \in I} \left(\sum_{n=m+1}^{m+p} |f_n(x)| \right) \\ &\leq \sum_{n=m+1}^{m+p} \sup_{x \in I} |f_n x| = \sum_{n=m+1}^{m+p} \|f_n\|\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \|(f_n)\| \text{ CV} \quad &\Longrightarrow \begin{cases} \forall \varepsilon > 0, \quad \exists m_\varepsilon \in \mathbb{N}, \quad \forall m, p \in \mathbb{N} \\ m \geq m_\varepsilon \implies \| \sum_{n=m+1}^{m+p} f_n \| \leq \sum_{n=m+1}^{m+p} \|f_n\| \leq \varepsilon \end{cases} \\ &\Longrightarrow \sum_{n=1}^{\infty} f_n \text{ CV Uniform on } I\end{aligned}$$

□

Remark. The inverse implication is not true, For instance

$$f_n(x) = \begin{cases} \frac{1}{n} & x = \frac{1}{n} \\ 0 & \text{if not} \end{cases} \qquad \text{on } [0, \infty)$$

$$\sum_{n=1}^{\infty} \|f_n\| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ DIV}$$

$$\begin{aligned}\text{But } \|S - S_m\| = \sup_{x \in [0,\infty)} \left| \sum_{n=m+1}^{\infty} f_n(x) \right| &= \begin{cases} 0 & \text{if } x \neq \frac{1}{k} \quad k \geq m+1 \\ \frac{1}{k} & \text{if } x = \frac{1}{k} \quad k \geq m+1 \end{cases} \\ &= \frac{1}{m+1} \to \infty \text{ as } m \to \infty\end{aligned}$$

Example

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2} \quad x \in \mathbb{R}$

$$f_n(x) = \frac{1}{n^2 + x^2} \qquad \|f_n\| = \frac{1}{n^2}$$

$$\sum_{n \geq 1} \frac{1}{n^2} \text{ CV} \quad \Longrightarrow \sum_{n \geq 1} \|f_n\| \text{ CV} \quad \Longrightarrow \sum_{n \geq 1} f_n \text{ CV uniform in } \mathbb{R}$$

3.3 Abel's Criterion for the uniform convergence

Theorem 3.3.1

Let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be two sequences of functions such that

1. $\exists M > 0$ such that $\|F_M\| \leq M \quad \forall m \in \mathbb{N}$ Where $F_m(x) = \sum_{n=1}^{m} f_n(x)$
2. $\sum \|g_{n+1} - g_n\|$ CV
3. $\lim_{n \to \infty} \|g_n\| = 0$

Then $\sum_{n=1}^{\infty} f_n g_n$ CV uniformaley on I

Example

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ $\quad D_c = (0, \infty)$ The series converge uniformaley on any interval of the form $[a, \infty)$ with $a > 0$

$$f_n(x) = (-1)^n \quad g_n(x) = \frac{1}{n^x} = \exp\left(-x \ln n\right)$$

$$\|\sum_{n=1}^{m} f_n\| \leq 1 \qquad \lim_{n\to\infty} \|g_n\| = \frac{1}{n^\alpha} \to 0$$

$$\begin{aligned}
\|g_{n+1} - g_n\| &= \sup_{x\geq 1}\left(g_{n+1} - g_n\right) = \sup_{x\geq 1}\left(\frac{1}{n^x} - \frac{1}{(n+1)^x}\right) \\
&= \sup_{x\geq 1} \frac{1}{n^x}\left(1 - \frac{1}{\left(1-\frac{1}{n}\right)^x}\right) = \sup_{x\geq 1}\frac{1}{n^x}\left(1 - \exp\left(-x\ln n\right)\right) \\
&\leq \sup_{x\geq a} \frac{x}{n^{(x+1)}} = \frac{a}{n^a + 1}
\end{aligned}$$

Since $a + 1 > 1$ so $\sum_{n=1}^{\infty} \|g_{n+1} - g_n\|$ Converge.

2. $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ on $[\pi/6, \pi/2]$:

$$f_n(x) = \sin(nx) \quad g_n(x) = \frac{1}{n}$$

$$\lim_{n\to\infty} \|g_n\| = 0$$

$$\|g_{n+1} - g_n\| = \|\frac{1}{n+1} - \frac{1}{n}\| \sim \frac{1}{n^2}$$

so $\sum_{n=1}^{\infty} \|g_{n+1} - g_n\|$ CV

$$\begin{aligned}
\left|\sum_{n=1}^{m} \sin(nx)\right| &= \left|\operatorname{Im}\left(\sum_{n=0}^{m} e^{inx}\right)\right| = \left|\operatorname{Im}\left(\sum_{n=0}^{m} (e^{ix})^n\right)\right| \\
&= \left|\operatorname{Im}\left(\frac{1 - e^{i(n+1)x}}{1 - e^{ix}}\right)\right| \\
&\leq \left|\frac{1 - e^{i(n+1)x}}{1 - e^{ix}}\right| \leq \frac{1 + \left|e^{i(n+1)x}\right|}{1 - e^{ix}} = \frac{2}{\sqrt{2(1-\cos x)}}
\end{aligned}$$

$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ CVU on $[\pi/6, \pi/2]$

3.4 Properties of the uniform convergence

In all this section we let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{F}(I, \mathbb{R})$

Theorem 3.4.1

Suppose that $\sum_{n=1}^{\infty} f_n$ uniformally converge and $(f_n)_{n\in\mathbb{N}}$ is continuous on I for all $n \geq 1$, then $\sum_{n=1}^{\infty} f_n$ is continuous on I

Proof. Let $S_m = \sum_{n=1}^{m} f_n$

$$\sum_{n=1}^{\infty} f_n \text{ CVU on } I \iff (S_m)_{m\in\mathbb{N}} \text{ CVU on } I$$

Since $(f_n)_{n\in\mathbb{N}}$ is continuous on $I \quad \forall n \geq 1$, we have (S_m) is continuous on I for all $n \in \mathbb{N}$, By Corollary 3.5 of chapter sequences of functions, we have :

$$S = \sum_{n=1}^{\infty} f_n \text{ is continuous on } I$$

□

Remark. If $\sum_{n=1}^{\infty} f_n$ UCV on I, and $\lim_{x\to a} f_n(x) = l_n \in \mathbb{R}$, with $a \in \overline{I}$, then :

$$\lim_{x\to a} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to a} f_n(x)$$

Theorem 3.4.2

If for all $n \in \mathbb{N}$, f_n is integrable on $[a,b] \subset I$, and $\sum_{n=1}^{\infty} f_n$ UCV on $[a,b]$, then $\sum_{n=1}^{\infty} f_n$ is integrable on $[a,b]$ and we have

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(t)\right) dt = \sum_{n=1}^{\infty} \int_a^b (f_n(t))\, dt$$

Example

Consider the series $\sum_{n=1}^{\infty}(-1)^n x^n, \quad I = [0,1]$
We apply abel's criterion :

$$f_n = (-1)^n \quad \|\sum_{n=1}^{m} f_n\| \leq 1 \quad \forall m \in \mathbb{N}$$

$$g_n(x) = \frac{x^n}{n} \quad \|g_n\| = \frac{1}{n} \to \infty$$

$$\begin{aligned}\|g_{n+1} - g_n\| &= \sup_{x\in[0,1]} \left|\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right| = \sup_{x\in[0,1]} x^n \left|\frac{1}{n} - \frac{x}{n+1}\right| \\ &\leq \sup_{x\in[0,1]} \left(\frac{1}{n} - \frac{x}{n+1}\right) = \sup_{x\in[0,1]} \left|\frac{n(1-x)+1}{n(n+1)}\right| \leq \frac{1}{n(n+1)} \sim \frac{1}{n^2}\end{aligned}$$

$\sum_{n=1}^{\infty} \|g_{n+1} - g_n\|$ CV

$$\begin{aligned}\int_0^1 \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n\right) dx &= \sum_{n=1}^{\infty} \left(\int_0^1 \frac{(-1)^n}{n} x^n\right) dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}\end{aligned}$$

Theorem 3.4.3 Differentiability

Suppose that $(f_n)_{n\in\mathbb{N}}$ is continuously differentiable on $[a,b] \subset I$, for all $n \geq 1$ and $\sum_{n=1}^{\infty} f_n(x)$ converge for some $x_0 \in [a,b]$, if $\sum_{n=1}^{\infty} f_n'$ UCV on $[a,b]$, then $\sum_{n=1}^{\infty} f_n$ UCV and it's sum is continuously differentiable on $[a,b]$, and we have :

$$\left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'$$

Example

1.
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ UCV on $[-a, a] \quad \forall a > 0.$

$$\sum_{n=0}^{\infty} \sup_{x \in [-a,a]} \frac{x^n}{n!} = \sum \frac{a^n}{n!} \text{ CV}$$

$f_n'(x) = \frac{x^{n-1}}{(n-1)!}$ if $n \geq 1$, $f_1'(x) = 1' = 0$

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ CV Normally on } [-a, a]$$

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ CVU on } [-a, a]$$

Therefore, $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

2. $S(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad D_c = \mathbb{R}$ Use d'Almbert

$$S'(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!}$$
$$S''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-2)!} x^{2n-2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!} = -S(x)$$

$$\begin{aligned} y'' + y &= 0 \\ S(x) &= y(x) = A\cos(x) + B\sin x \\ S(0) &= 1 \quad y(0) = A \implies A = 1 \\ S'(0) &= 0 \quad y'(0) = -A\sin(x) + B\cos(x) \implies B = 0 \end{aligned}$$

$$\text{Hence } S(x) = \cos x$$
$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

3.5 Abel's Criterion for the uniform convergence

Example

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^x}$ CVU on $[a, \infty)$ for $a > 0$.

$$\|\frac{1}{n^x}\| = \frac{1}{n^a} \to 0 \quad \left|\sum_{n=1}^{\infty} \frac{(-1)^n}{f_n}\right| < 1$$

$$\sum_{n=1}^{\infty} \|g_n - g_{n+1}\| = \sum_{n=1}^{\infty} \sup_{x\in[a,\infty)} \left(\frac{1}{n^x} - \frac{1}{(n+1)^x}\right)$$

$$\frac{1}{n^x} - \frac{1}{(n+1)^x} = \frac{1}{n^x}\left(1 - \frac{1}{(1+n)^x}\right) = \frac{1}{n^x} = \frac{1}{n^x} = \frac{1}{n^x}\left(1 - \exp\left(-x\ln\left(1 - \frac{1}{n}\right)\right)\right)$$

For $0 < a < 1 < b$

$$\|g_n - g_{n+1}\| \leq max\left(\sup_{x\in[a,b]}\left(\frac{1}{n^x} - \frac{1}{(n+1)^x}\right), \sup_{x\in[b,\infty)}\left(\frac{1}{n^x} - \frac{1}{(n+1)^x}\right)\right)$$

$$\sup_{x\in[a,b]} \frac{1}{n^x}\left(1 - \frac{1}{\left(1+\frac{1}{n}\right)^x}\right) \leq \frac{1}{n^a}\left(1 - \exp\left(-b\ln\left(1 + \frac{1}{n}\right)\right)\right) \sim \frac{1}{n^a}\frac{b}{n} = \frac{b}{n^{a+1}}$$

$$\sup_{x\in[b,\infty)} (\text{ something }) \leq \frac{1}{n^b}$$

$$\sum_{n=1}^{\infty} \frac{b}{n^{a+1}} \text{ CV} \quad a+1 > 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n^b} \text{ CV} \quad b > 1$$

Chapter 4

Power Series

4.1 Basic facts of complex analysis

Let $a \in \mathbb{C}$ and r in $[0, \infty]$
The open disk center at a of radius r, the set $\mathcal{D}(a, r)$ defined by

$$\mathcal{D}(a, r) = \{z \in \mathbb{C} : \quad |z - a| < r\}$$

The closed disk centered at a of radius r is the set $\overline{\mathcal{D}(a, r)}$ defined by

$$\overline{\mathcal{D}(a, r)} = \{z \in \mathbb{C} : \quad |z - a| \leq r\}$$

If $r = \infty$, then $\mathcal{D}(a, \infty) = \overline{\mathcal{D}}(a, \infty) = \mathbb{C}$
Let $(z_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers, we say that $(z_n)_{n \in \mathbb{N}}$ converges to $l \in \mathbb{C}$ and we write $\lim_{n \to \infty} z_n = l$, if

$$\forall \varepsilon > 0 \quad \forall n \in \mathbb{N} \text{ s.t. } n \geq N \implies |z_n - l| \leq \varepsilon$$

We say $(z_n)_{n \in \mathbb{N}}$ is a cauchy sequence if for all

$$\forall \varepsilon > 0 \quad \exists n \in \mathbb{N} \text{ s.t. } \forall n, m \in \mathbb{N} \quad m, n \geq \mathbb{N} \implies |z_m - z_n| \leq \varepsilon$$

Since for any $z = x + iy$, we have $\max(|x|, |y|) \leq |z| = \sqrt{x^2 + y^2} \leq |x| + |y|$ we conclude that $z_n = x_n + iy_n$ is of cauchy if and only if $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are cauchy.

$$\text{Therefore, } (z_n)_{n \in \mathbb{N}} \text{ is of cauchy } \iff (z_n)_{n \in \mathbb{N}} \text{ is convergent}$$

Let Ω be a open set in $\mathbb{C}$ and let $f : \Omega \longrightarrow \mathbb{C}$ be function, and let $a \in \overline{\Omega}$ adherence, the function f is said to

1. Have a limit

$$\lim_{z \to a} f(x) = l \iff \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } \forall z \in \Omega \implies 0 < |z - a| \leq \delta \implies |f(z) - l| \leq \varepsilon$$

2. Be a continuous at a if $\exists r > 0$ such that $\overline{\mathcal{D}(a, r)} \subset \Omega$ and $\lim_{z \to a} f(z) = f(a)$

3. Be continuous on Ω, if its continuous at every point Ω

4. Differentiable at a if has derivative equals to $f'(a)$, if $\exists r > 0$ such that

$$\mathcal{D} \subset \Omega \text{ and } f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

5. Differentiable on Ω (Holomorph) if it's at every point of Ω

6. Have a primitive on Ω if $\exists F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$
7. Be of class $\mathcal{C}^k$ on Ω, if for all $i \in \{0, 1, \dots, (k-1)\}$ $f^{(i)}$ is differentiable and $f^{(i+1)} = \left(f^{(i)}\right)'$ and $f^{(k)}$ is continuous on Ω, we write $f \in \mathcal{C}^k(\Omega)$
8. Be $\mathcal{C}^\infty$ on Ω if $f \in \bigcap_{k \geq 0} \mathcal{C}^k(\Omega)$

Example

$$f(z) = z^n \quad n \in \mathbb{N} \quad f'(z) = nz^{n-1}$$

Remark. You will see, that if f is holomorph on Ω then f is $\mathcal{C}^\infty$ on Ω

4.2 Power Series

Definition 4.2.1

We call a power series centered at z_0 any series of functions, having the form $\sum_{n=1}^{\infty} a_n(z - z_0)^n$, where (a_n) is a sequence of complex numbers, and for all $n \in \mathbb{N}$, a_n is the coefficient of order n

Example

1. All polynomials functions are power series
2. The geometric series $\sum_{n=1}^{\infty} z^n$ is a power series.

Theorem 4.2.1 First Abel's lemma

Let $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ be a power series and let $z_1 \in \mathbb{C}$, if $\sum_{n=1}^{\infty} a_n(z_1 - z_0)^n$ converges, then $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ converges absolutely for all $z \in \mathcal{D}(z_0, |z_1 - z_0|)$

Proof.

$$\begin{aligned} \sum_{n=1}^{\infty} a_n(z - z_0)^n &\implies \lim_{n \to \infty} a_n(z_1 - z_0)^n = 0 \\ &\implies \exists M > 0 \text{ s.t. } |a_n(z_1 - z_0)^n| \leq M \quad \forall n \in \mathbb{N} \end{aligned}$$

For $z \in \mathcal{D}(z_0, |z_1 - z_0|)$, we have $|z - z_0| < |z_1 - z_0|$, then

$$\sum_{n=1}^{\infty} |a_n(z - z_0)^n| = \sum_{n=1}^{\infty} |a_n|\, |z_1 - z_0|^n \left(\frac{|z - z_0|}{|z_1 - z_0|}\right)^n \leq M \sum_{n=1}^{\infty} \left(\frac{|z - z_0|}{z_1 - z_0}\right)^n \text{ CV}$$

□

Corollary 4.2.2

Let $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ be a power series and let $z_1 \in \mathbb{C}$, if $\sum_{n=1}^{\infty} a_n(z_1 - z_0)^n$ diverges, then $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ diverges for all $z \in \{\alpha \in \mathbb{R} : |z - z_0| > |z_1 - z_0|\}$

Proof. If $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ CV for some $z = z_2 \in \mathbb{C}$ with $|z - z_0| > |z_1 - z_0|$, then from above $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ CV $\forall z \in \mathcal{D}(z_0, |z_2 - z_0|)$, this is impossible since $z_1 \in \mathcal{D}(z_0, |z_2 - z_0|)$ □

Theorem 4.2.3

Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$, be a power series and let $R>0$, such that the series converges for all $z \in \mathcal{D}(z_0, R)$, then for all $r \in (0, R)$ the series converges normally in the disk $\overline{\mathcal{D}}(z_0, r)$

Proof. For all $z \in \mathcal{D}(z_0, r)$, we have

$$\sum_{n=1}^{\infty} \left(\sup_{z \in \overline{\mathcal{D}}(z_0,r)} |a_n(z-z_0)^n| \right) \leq \sum_{n=1}^{\infty} |a_n| \, |z_2 - z_0|^n \text{ CV}$$

where $z_2 \in \mathbb{C}$, with $r < |z_2 - z_0| = R_1 < R_2$ □

Definition 4.2.2

Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series, and let $\mathcal{D}_C$ denotes it's domain of convergence, we call radius of convergence of the series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$,

$$\mathcal{R} = \begin{cases} \sup D^* & \text{if } D^* \text{ is bounded} \\ \infty & \text{if not} \end{cases}$$

Where $D^* = \{|z-z_0|, z \in D_c\}$, where $D_c = \{z \in \mathbb{C} : \sum_{n=1}^{\infty} a_n(z-z_0)^n \text{ CV }\}$

Remark. The disk $\mathcal{D}(z_0, R)$ is called the open disk of convergence

Example

1. $\sum_{n=1}^{\infty} z^n \quad \mathcal{D}_c = \mathcal{D}(0,1) = \{z \in \mathbb{C} : |z| < 1\}$,$\mathcal{R} = \sup\{|z| : z \in \mathcal{D}(0,1)\} = 1$
2. $\sum_{n=1}^{\infty} \frac{z^n}{n!} \quad D_c = \mathbb{C} \implies \mathcal{R} = \infty$

Remark. If $\mathcal{R}$ is the radius of convergence of the series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$, we haven't $\mathcal{D}_c = \mathcal{D}(z_0, \mathcal{R})$

Example

1. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, set $U_n(x) = \left|\frac{x^n}{n^2}\right| = \frac{|x|^n}{n^2}$

$$\frac{U_{n+1}(x)}{U_n(x)} = |x| \left(\frac{n}{n+1}\right)^2 \to |x|$$

- if $|x| < 1$, then $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ CV (D'Almbert)
- if $|x| = 1$, $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} = \frac{1}{n^2}, \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges.
- if $|x > 1|$, $\lim_{n\to\infty} \frac{x^n}{n^2} = \infty \implies \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ DIV
- if $x < -1$, $\lim_{n\to\infty} \frac{x^{2n}}{4n^2} = \infty \implies \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ DIV

Domain of convergence $\mathcal{D}_c = [-1, 1]$ and $\mathcal{R} = 1$ which is the sup of the $\mathcal{D}_c$, Note : Radius of convergence excludes the boundarys!, check definition again.

Theorem 4.2.4

Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series having $\mathcal{R}$ as a radiuos of convergence, the following assertion holds

- $\mathcal{R} = 0 \iff \mathcal{D} = \{z_0\}$
- $\mathcal{R} = \infty \iff \mathcal{D} = \mathbb{C}$
- $\mathcal{R} \in (0,\infty)$, then :

$$\begin{cases} |z-z_0| < \mathcal{R} \implies \sum_{n=1}^{\infty} |a_n(z-z_0)^n| \text{ CV} \\ |z-z_0| > \mathcal{R} \implies \sum_{n=1}^{\infty} a_n(z-z_0)^n \text{ DIV} \end{cases}$$

Proof. 1.

$$\mathcal{D}_c = \{z_0\} \implies \mathcal{R} = 0$$

$$\mathcal{R} = 0 \implies \mathcal{D}_c = \{z_0\}$$

Indeed if there is $z_1 \neq z_0$ such that $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ CV, then by above Theorem 4.1.3

$$\sum_{n=1}^{\infty} a_n(z-z_0)^n \text{ CV } \forall z \in \mathcal{D}(z_0, |z_1-z_0|)$$

Hence $\mathcal{D}(z_0, |z_1-z_0|) \subset \mathcal{D}_c$ and $0 = \mathcal{R} > |z_1-z_0| > 0$, Contradiction.

2.

$$\mathcal{D}_c = \mathbb{C} \implies \mathcal{R} = \infty \text{ is clear}$$

If $\mathcal{R} = \infty$, then $D_c = \mathbb{C}$, if there is a point $z \in \mathbb{C}$, such that $\sum_{n=1}^{\infty} a_n(z_1-z_0)^n$ DIV, then by Corollary 4.1.4, $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ DIV for all $z \in \mathbb{C}$, with $|z-z_0| > |z_1-z_0|$ this implies that $\mathcal{D}_c \subset \mathcal{D}(z_0, |z_1-z_0|)$, this contradicts the fact that $\mathcal{R} = \infty$

3. $\mathcal{R} \in (0,\infty)$, let $a \in \mathcal{D}(z_0,\mathcal{R})$, we have $|a-z_0| < \mathcal{R}$, there is $b \in \mathbb{C}$ such that

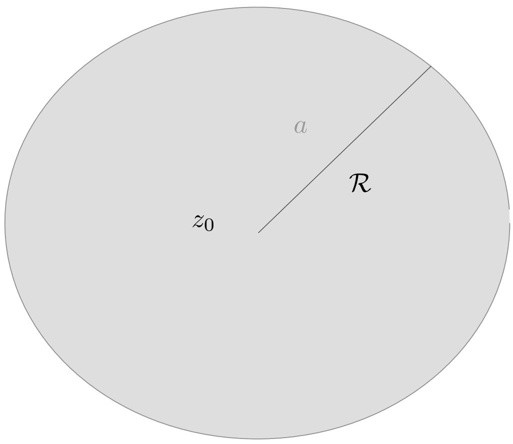

Figure 4.1: draw

$|a-z_0| < |b-z_0| < \mathcal{R}$ and $\sum_{n=1}^{\infty} a_n(b-z_0)^n$ CV
By Theorem 4.1.3, $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ CV for all $z \in \mathcal{D}(z_0, |b-z_0|)$ since $a \in \mathcal{D}(z_0, |b-z_0|)$, the series $\sum_{n=1}^{\infty} |a_n(a-z_0)^n|$ CV.
($\impliedby$) Let $a \in \mathbb{C}$, such that $|a-z_0| > \mathcal{R}$, if $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ CV then (By Theorem 4.1.3), we have $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ CV, for all $z \in \mathcal{D}(z_0, |a-z_0|)$ with $|a-z_0| > \mathcal{R}$, Contradiction!, with the definition of $\mathcal{R}$.

□

Theorem 4.2.5

Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series with a radius of convergence equal to $\mathcal{R}$, Let

$$\Omega_1 = \left\{ |z-z_0| : \left(a_n(z-z_0)^n\right)_{n\geq 0} \text{ is Bounded} \right\}$$
$$\Omega_2 = \left\{ |z-z_0| : \left(a_n(z-z_0)^n\right)_{n\geq 0} \text{ is Unbounded} \right\}$$

Then either $\mathcal{R} = \infty$ or Ω_1 is upper bounded, and Ω_2 is lower bounded and we have

$$\mathcal{R} = \sup \Omega_1 = \inf \Omega_2$$